An Analytical Approach to 3D Sound Field Reproduction Employing Spherical Distributions of Non-Omnidirectional Loudspeakers

Jens Ahrens and Sascha Spors
Deutsche Telekom Laboratories, Technische Universität Berlin
Ernst-Reuter-Platz 7, 10587 Berlin, Germany
Email: {jens.ahrens,sascha.spors}@telekom.de

***Abstract*—We present an approach targeting the physical reproduction of sound fields by means of spherical distributions of non-omnidirectional loudspeakers. The focus of this paper lies on the modal incorporation of the loudspeaker's spatio-temporal transfer function into the loudspeaker driving function.**

I. Introduction

Traditionally, massive-multichannel sound field reproduction approaches like wave field synthesis or higher order Ambisonics assume that the involved secondary sources (i.e. loudspeakers) are omnidirectional. For lower frequencies, this assumption is indeed approximately fulfilled when conventional loudspeakers with closed cabinets are considered. However, for higher frequencies above a few thousand Hertz complex radiation patterns evolve.

A number of approaches based on the theory of multiple-input-multiple-output (MIMO) systems have been proposed in order to compensate for the influence of the reproduction room and the loudspeaker radiation characteristics, e.g. [1], [2], [3]. Room compensation (which includes compensation of loudspeaker radiation properties) requires realtime analysis of the reproduced sound field and adaptive algorithms due to the time-variance of room acoustics (e.g. temperature variations). Compensation of the loudspeaker radiation characteristics, such as directivity and frequency response, is less complex since it can be assumed that these characteristics are time-invariant. No adaptation and therefore no real-time analysis is required. However, in order that the radiation characteristics can be compensated for while neglecting the reproduction room, the radiation characteristics of the entire secondary source setup have to be measured under anechoic conditions. When certain physical constraints are accepted, a significant reduction of complexity can be achieved and a continuous formulation can be established. Besides time-invariance, the fundamental physical constraints introduced in the presented approach are:

(1) The secondary source arrangement is spherical.
(2) The spatio-temporal transfer function of the secondary sources is shift-invariant with respect to rotation around the center of the secondary source distribution. In other words, all individual loudspeakers have to exhibit equal radiation characteristics and have to be orientated towards the center of the secondary source setup.

Requirement (1) can obviously be fulfilled. Preliminary measurements undertaken at Deutsche Telekom Laboratories have shown that typical commercially available loudspeakers with closed cabinets indeed exhibit similar to equal spatio-temporal transfer functions in anechoic condition and when only one model of loudspeakers is considered. This suggests that requirement (2) can also be fulfilled when the acoustical properties of the reproduction room are ignored.

We emphasize that the presented approach is not a compensation for deviations of the loudspeaker radiation characteristics from certain assumptions (e.g. omnidirectionality). It is rather such that the formulation of the approach allows for an explicit consideration thereof. For convenience, we use the term *directivity filter* to refer to that component of the secondary source driving function which represents the spatio-temporal transfer function of the secondary sources.

The approach treated in this paper has been presented by the authors in [4], [5], whereby formulations were kept general. The 2.5-dimensional formulation of the problem can be found in [6]. In this contribution, we investigate in detail the properties of the loudspeaker directivity filters which arise when spherical secondary source setups are employed for three-dimensional reproduction.

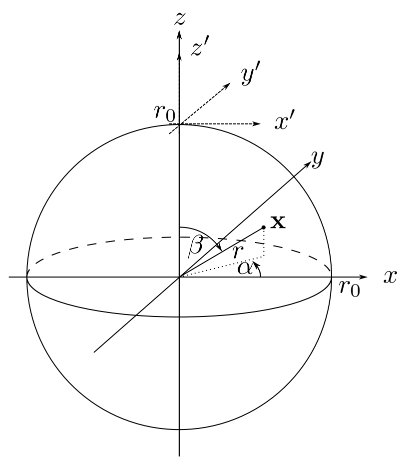


Fig. 1. The coordinate systems used in this paper. Primed quantities belong to a local coordinate system at position $\mathbf{x'}_0 = [0\ 0\ r_0]^T$ (see text). The sphere indicates the secondary source distribution.

Ahrens, J.; Spors, S., "An analytical approach to 3D sound field reproduction employing spherical distributions of non-omnidirectional loudspeakers," Communications, Control and Signal Processing (ISCCSP), 2010 4th International Symposium on, 3-5 March 2010, p. 1 - 5, DOI: 10.1109/ISCCSP.2010.5463364

© 2010 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works.

II. Nomenclature and Mathematical Preliminaries

The following notational conventions are used: Vectors are denoted by lower case boldface. The three-dimensional position vector in Cartesian coordinates is given as $\mathbf{x} = [x\ y\ z]^T$. The Cartesian coordinates are linked to the spherical coordinates via $x = r\cos\alpha\sin\beta$, $y = r\sin\alpha\sin\beta$, and $z = r\cos\beta$. α denotes the azimuth, β the zenith angle. Refer also to Fig. 1.

The acoustic wavenumber is denoted by k. It is related to the temporal frequency by $k^2 = \left(\frac{\omega}{c}\right)^2$ with ω denoting the radial frequency and c the speed of sound. Outgoing spherical waves are denoted by $\frac{1}{r}e^{-i\frac{\omega}{c}r}$. i is the imaginary unit ($i = \sqrt{-1}$). We employ a number of standard mathematical tools which are defined below.

A sound field can be described by its spherical harmonics expansion coefficients $\mathring{F}_n^m(r,\omega)$ as [7]

$$F(\mathbf{x},\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathring{F}_n^m(r,\omega)\ Y_n^m(\alpha,\beta)\ . \tag{1}$$

The spherical harmonics $Y_n^m(\alpha,\beta)$ are defined as

$$Y_n^m(\alpha,\beta) = \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}} \cdot P_n^m(\cos\beta)\cdot e^{im\alpha}\ , \tag{2}$$

with $P_n^m(\cdot)$ denoting the m-th order associated Legendre polynomial of n-th degree.

Alternatively to the spherical harmonics expansion coefficients $\mathring{F}_n^m(r,\omega)$, a sound field can be described by the coefficients $\breve{F}_n^m(\omega)$ as

$$F(\mathbf{x},\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \breve{F}_n^m(\omega) B_n^m(\mathbf{x},\omega)\ , \tag{3}$$

whereby the basis $B_n^m(\mathbf{x},\omega)$ is the singular basis $S_n^m(\mathbf{x},\omega)$ for purely diverging wave fields, or the regular basis $R_n^m(\mathbf{x},\omega)$ for source-free wave fields. Explicitly,

$$S_n^m(\mathbf{x},\omega) = h_n^{(2)}\left(\frac{\omega}{c}r\right) Y_n^m(\alpha,\beta)\ , \tag{4}$$

$$R_n^m(\mathbf{x},\omega) = j_n\left(\frac{\omega}{c}r\right) Y_n^m(\alpha,\beta)\ . \tag{5}$$

$h_n^{(2)}\left(\frac{\omega}{c}r\right)$ denotes the n-th order spherical Hankel function of second kind, $j_n\left(\frac{\omega}{c}r\right)$ the n-th order spherical Bessel function of first kind [7].

The relation between the coefficients $\mathring{F}_n^m(r,\omega)$ and $\breve{F}_n^m(\omega)$ can be deduced from (1), (3), (4), and (5).

III. Derivation of the Driving Function

The approach outlined by the authors in [4] constitutes the *single-layer potential* solution to the problem of sound field reproduction. For a spherical distribution of secondary sources and with radius r_0 centered around the origin of the coordinate system (refer to Fig. 1) the single-layer potential is given by [4]

$$P(\mathbf{x},\omega) = \int_0^{2\pi}\int_0^{\pi} D(\mathbf{x}_0,\omega)\ G(\mathbf{x}-\mathbf{x}_0,\omega)\ r_0\ \sin\beta_0 d\beta_0 d\alpha_0\ , \tag{6}$$

whereby $P(\mathbf{x},\omega)$ denotes the reproduced sound field, $D(\mathbf{x}_0,\omega)$ the driving signal of the secondary source located at the position $\mathbf{x}_0 = r_0 \cdot [\cos\alpha_0\sin\beta_0\ \sin\alpha_0\sin\beta_0\ \cos\beta_0]^T$, and $G(\mathbf{x}-\mathbf{x}_0,\omega)$ its spatio-temporal transfer function. Eq. (6) is also termed *reproduction equation*.

A fundamental property of (6) is its inherent non-uniqueness and ill-posedness [4]. I.e. in certain situations, the solution is undefined and so-called *critical* or *forbidden frequencies* arise. The forbidden frequencies are discrete and represent the resonances of the cavity under consideration. However, there are indications that the forbidden frequencies are only of minor relevance when practical implementations are considered [7].

Note that we assume $G(\cdot)$ to be rotation invariant with respect to rotation around the origin of the coordinate system (we write $G(\mathbf{x}-\mathbf{x}_0,\omega)$ instead of $G(\mathbf{x}|\mathbf{x}_0,\omega)$) [7]. This requires that all secondary sources have to exhibit equal spatio-temporal characteristics and have to be orientated towards the center of the secondary source distribution.

Equation (6) can be interpreted as a convolution along the surface of a sphere. In that case, the convolution theorem

$$\mathring{P}_n^m(r,\omega) = 2\pi r_0 \sqrt{\frac{4\pi}{2n+1}}\ \mathring{D}_n^m(\omega)\cdot \mathring{G}_n^0(r,\omega)\ , \tag{7}$$

applies [8] which can be reformulated as [4]

$$\mathring{D}_n^m(\omega) = \frac{1}{2\pi r_0}\sqrt{\frac{2n+1}{4\pi}}\ \frac{\breve{P}_n^m(\omega)}{\breve{G}_n^0(\omega)}\ . \tag{8}$$

The asymmetry of the convolution theorem (7), $\mathring{P}_n^m(r,\omega)$ vs. $\mathring{G}_n^0(r,\omega)$ is a consequence of the definition of (6) as left convolution. An according convolution theorem for right convolutions exists [8]. Note that (7) is the analog to the mode-matching which is performed in the traditional Ambisonics approach.

Combining (8) with (1) yields the secondary source driving function $D(\alpha,\beta,\omega)$ for the reproduction of a desired sound field with expansion coefficients $\breve{P}_n^m(\omega)$ as

$$D(\alpha,\beta,\omega) = \frac{1}{2\pi r_0}\times \times\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sqrt{\frac{2n+1}{4\pi}}\frac{\breve{P}_n^m(\omega)}{\breve{G}_n^0(\omega)} Y_n^m(\alpha,\beta)\ . \tag{9}$$

Equation (9) can be verified by inserting it into (6). After interchanging the order of integration and summation and exploitation of the orthogonality of the spherical harmonics, one arrives at the desired sound field, thus proving perfect reproduction.

However, equation (9) generally only holds for $|\mathbf{x}| < r_0$ due to the fact that the coefficients $\breve{P}_n^m(\omega)$ and $\breve{G}_n^0(\omega)$ are typically derived from interior expansions [7].

IV. Incorporation of the Secondary Source Directivity Coefficients

The coefficients $\breve{G}_n^0(\omega)$ apparent in the driving function (9) describe the spatio-temporal transfer function of a secondary source which is positioned at the north pole of the sphere

($\mathbf{x}_0 = [0\ 0\ r_0]^T$) and orientated towards the center of the coordinate system (which coincides with the center of the secondary source distribution). The expansion center is the origin of the coordinate origin. This follows directly from the convolution theorem (7).

However, typical loudspeaker directivity measurements such as [9] yield the coefficients $\breve{G'}_{n'}^{m'}(\omega)$ (see below) of an expansion of the loudspeaker's spatio-temporal transfer function around the acoustical center of the loudspeaker denoted by $\mathbf{x'}_0$. The acoustical center of a loudspeaker is referred to as the position of the latter in the remainder. For convenience, we assume in the following that the loudspeaker under consideration is positioned at $\mathbf{x'}_0 = \mathbf{x}_0 = [0\ 0\ r_0]^T$ and is orientated towards the origin of the global coordinate system. We term the coefficients $\breve{G'}_{n'}^{m'}(\omega)$ *secondary source directivity coefficients*.

We establish a local coordinate system with origin at $\mathbf{x}_0$ and whose axes are parallel to those of the global origin (refer to Fig. 1). Then the spatio-temporal transfer function $G(\mathbf{x'},\omega)$ of the considered loudspeaker can be described as

$$G(\mathbf{x'},\omega) = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \breve{G'}_{n'}^{m'}(\omega)\ S_{n'}^{m'}(\mathbf{x'},\omega) \tag{10}$$

with respect to the local coordinate system. Note that

$$\mathbf{x'} = \mathbf{x'}(\mathbf{x}) = \mathbf{x} + \Delta\mathbf{x} = \mathbf{x} - r_0 \mathbf{e}_z\ , \tag{11}$$

with $\Delta\mathbf{x} = [0\ 0\ -r_0]$. $\mathbf{e}_z$ denotes the unit vector pointing into positive z-direction.

In the remainder of this section we demonstrate how the coefficients $\breve{G}_n^0(\omega)$ required by the secondary source driving function (9) can be yielded from the directivity coefficients $\breve{G'}_{n'}^{m'}(\omega)$ by applying appropriate translation operations.

The required translation along the z-axis in negative direction can be performed elegantly by 1) flipping the z-axis, 2) following the new axis by r_0 in positive direction, and 3) flipping back the new axis to coincide again with the z-axis [10].

Flipping the z-axis negates all odd harmonics, i.e. $Y_n^m(\alpha, \pi - \beta) = (-1)^{n+m} Y_n^m(\alpha,\beta)$ [11]. The translation of the singular part $S_{n'}^{m'}(\mathbf{x'},\omega)$ of the expansion (10) along the z-axis in positive direction results in [10]

$$S_{n'}^{m'}(\mathbf{x} + r_0\mathbf{e}_z,\omega) = \sum_{n=|m'|}^{\infty} (S|R)_{n\,n'}^{m'}(r_0,\omega)\ R_n^{m'}(\mathbf{x})\ . \tag{12}$$

The notation $(S|R)$ indicates that the translation represents a change from a singular basis expansion to a regular basis expansion.

Inserting (12) in (10), flipping there and back the z-axis, and re-ordering of the sums reveals the general form of $\breve{G}_n^m(\omega)$ as

$$G(\mathbf{x},\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} R_n^m(\mathbf{x}) \times \underbrace{\times \sum_{n'=|m|}^{\infty} \breve{G'}_{n'}^{m}(\omega)\ (-1)^{n+n'+2m}\ (S|R)_{n\,n'}^{m}(r_0,\omega)}_{=\ \breve{G}_n^m(\omega)}\ . \tag{13}$$

Note that we replaced m' with m in (13) for convenience.

From the driving function (9) we can deduce that we do not need all coefficients $\breve{G}_n^m(\omega)$ but only $\breve{G}_n^0(\omega)$

$$\breve{G}_n^0(\omega) = \sum_{n'=|m|}^{\infty} \breve{G'}_{n'}^{0}(\omega)\ (-1)^{n+n'}\ (S|R)_{n\,n'}^{0}(r_0,\omega)\ . \tag{14}$$

This reveals that only the subset $\breve{G'}_{n'}^{0}(\omega)$ of the secondary source directivity coefficients $\breve{G'}_{n'}^{m'}(\omega)$ need to be known. The former represent those modes of $G(\mathbf{x'},\omega)$ which symmetric with respect to rotation around the vertical axis through the expansion center.

This fact further facilitates the translation significantly. The required *zonal* translation coefficients can be computed from combinations of the initial values [10]

$$(S|R)_{n\,0}^{0}(r_0,\omega) = (-1)^n \sqrt{2n+1}\ h_n^{(2)}\left(\frac{\omega}{c} r_0\right) \tag{15}$$

$$(S|R)_{0\,n'}^{0}(r_0,\omega) = \sqrt{2n'+1}\ h_{n'}^{(2)}\left(\frac{\omega}{c} r_0\right) \tag{16}$$

via the recursion formula

$$a_{n'-1}(S|R)_{n\,n'-1}^{0}(r_0,\omega) - a_{n'}(S|R)_{n\,n'+1}^{0}(r_0,\omega) = a_n(S|R)_{n+1\,n'}^{0}(r_0,\omega) - a_{n-1}(S|R)_{n-1\,n'}^{0}(r_0,\omega)\ , \tag{17}$$

with

$$a_n = \frac{n+1}{\sqrt{(2n+1)(2n+3)}}\ . \tag{18}$$

It can be shown that the zonal translation coefficients are of the form

$$(S|R)_{nn'}^{0}(r_0,\omega) = \sum_{l'=0}^{n'} c^{l',n,n'} h_{n+2l'-n'}^{(2)}\left(\frac{\omega}{c} r_0\right)\ , \tag{19}$$

whereby $c^{l',n,n'}$ is a real number derived from (17) and (18).

V. Properties of the Directivity Filter

A. General

As evident from (14) and (19), each mode (n,m) of the directivity filter is given by a summation over the product of the secondary source directivity coefficient and the translation coefficient. The translation coefficients can be implemented via infinite impulse response filter (IIR) design approaches such as performed in [12]. Alternatively, the digital implementation can be obtained via an appropriate sampling of the analytical mathematical expression (14) which results then in a finite impulse response (FIR) representation.

Due to the fact that the secondary source directivity coefficients are typically yielded from measurements and are modeled as FIR filters, e.g. [9], we propose to also apply the FIR approach on the translation coefficients.

In order that the driving function (9) is defined neither mode $\breve{G}_n^0(\omega)$ of the directivity filter may exhibit zeros. From (14) it can be seen that each mode of the directivity filter is given by a summation over all directivity coefficients $\breve{G}'^{0}_{n'}(\omega)$ multiplied by the respective translation coefficient $(S|R)^{0}_{n\,n'}(r_0,\omega)$. The translation coefficients are linear combinations of spherical Hankel functions of the same argument but of different orders (refer to (19)). Spherical Hankel functions of different orders are linearly independent [7]. Thus, since spherical Hankel functions do not exhibit zeros, a linear combination of spherical Hankel functions and therefore also the translation coefficients do not exhibit zeros either. The fact whether the directivity filters (14) are defined or not is essentially dependent on the properties of the secondary source directivity coefficients $\breve{G}'^{0}_{n'}(\omega)$.

Secondary source directivity coefficients yielded from measurements of real loudspeakers do not per se result in a well-behaved driving function. Therefore (preferably frequency dependent) regularization has to be applied in order to yield a realizable solution. Contrary to conventional multichannel regularization, the presented approach allows for independent regularization of each mode n of the directivity filter. Thereby, stable modes need not be regularized while the regularization of individual unstable modes can be assumed to be favorable compared to conventional regularization of the entire filter.

B. Causality

Spherical Hankel functions of second kind are explicitly given by [10]

$$h_{n'}^{(2)}\left(\frac{\omega}{c}r_0\right) = i^{\,n'+1}\frac{e^{-i\frac{\omega}{c}r_0}}{\frac{\omega}{c}r_0}\sum_{f'=0}^{n'}\frac{(n'+f')!}{f'(n'-f')!}\left(\frac{1}{2i\frac{\omega}{c}r_0}\right)^{f'} . \quad (20)$$

The exponential term in (20) is independent of the order n' and can be factored out in (19). The exponential term represents a delay in time domain whose duration equals the propagation duration from a secondary source to the center of the secondary source contour. Since the exponential term appears in the numerator of the driving function (9) it turns into an anticipation. In order that the driving function stays causal, this anticipation has to be compensated by an appropriate pre-delay.

Furthermore, the secondary source directivity coefficients $\breve{G}'^{0}_{n'}(\omega)$ are generally not minimum phase. The inversion then leads to a filter of infinite length which can not be implemented with an FIR approach. A lack of the minimum phase property can also result in acausal components of the inverse filter. These have to be compensated for via a *modeling delay*. Such a modeling delay is simply an additional delay imposed on the driving function in order to make acausal components causal. Alternatively, the secondary source directivity coefficients can be approximated by minimum phase filters.

VI. Results

In order to illustrate the general properties of the presented approach we consider in the following a spherical distribution of highly directional secondary sources whose spatio-temporal transfer function is given by

$$\breve{G}'^{m'}_{n'}(\omega) = \begin{cases} \frac{i^{-n'}(N'-1)!N'!}{(N'+n')!(N'-n'-1)!} Y_{n'}^{m'}(0,0)^* & \forall n' \leq N'-1 \\ 0 & \text{elsewhere} \end{cases} . \quad (21)$$

with $N' = 13$. The asterisk $*$ indicates complex conjugation. The normalized far-field directivity of $G(\cdot)$ is depicted in Fig. 2.

Figure 3(a) depicts a continuous spherical distribution of

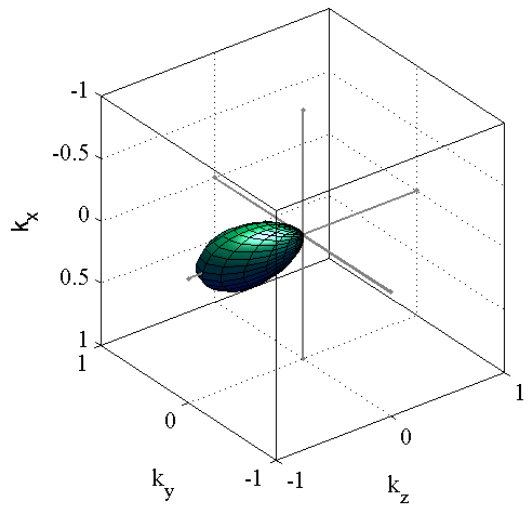
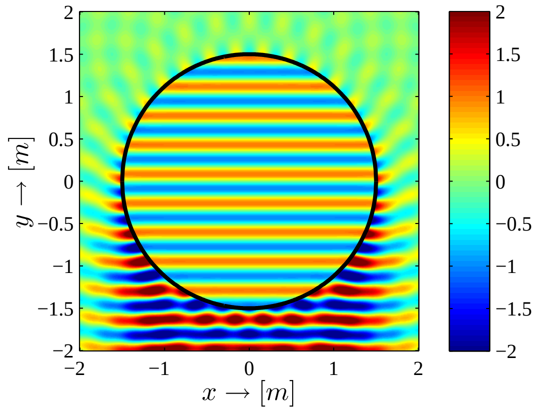


Fig. 2. Normalized far-field directivity of the secondary sources employed in Fig. 3(a).

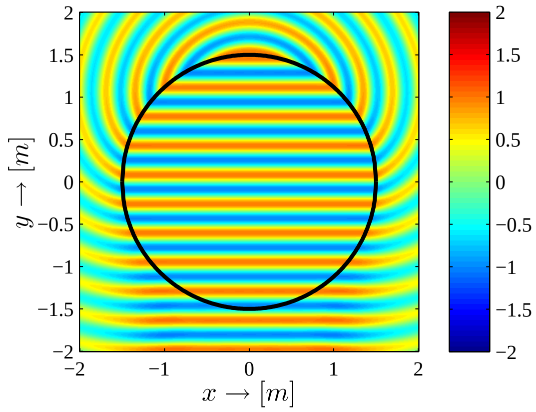
secondary sources with a directivity given by (21) reproducing a virtual plane wave of $f_{\mathrm{pw}} = 1000$ Hz. For comparison, Fig. 3(b) depicts the same virtual sound field reproduced by a continuous distribution of secondary monopoles which constitutes the classical scenario. It can be seen that, as theoretically predicted, the virtual sound field is perfectly reproduced inside the secondary source distribution, outside the secondary source distribution the reproduced sound fields differ considerably.

VII. Conclusions

An approach for sound field reproduction employing spherical arrangements of secondary sources was presented. It was focused on the general properties of the resulting secondary source driving function when non-omnidirectional secondary sources are used. In order that the presented approach is applicable the spatio-temporal characteristics of the employed secondary sources have to be invariant with respect to rotation around the center of the secondary source arrangement. In other words, all secondary sources have to exhibit equal radiation characteristics and have to be orientated towards the center of the secondary source arrangement.



(a) Secondary sources exhibiting a transfer function given by (21).



(b) Secondary monopoles.

Fig. 3. A virtual plane wave of unit amplitude and of frequency $f_{\mathrm{pw}} = 1000$ Hz reproduced by continuous distributions of secondary sources with different directivities. A cross-section through the horizontal plane is depicted.

Preliminary measurements of the ELAC 301 loudspeakers which are employed in the loudspeaker system installed at Deutsche Telekom Laboratories indicate that only very little variation in the spatio-temporal characteristics in apparent within different loudspeakers of the same model. This indicates that the presented approach is indeed applicable when all loudspeakers are of the same model. However, the investigation of resulting errors when such variation in the spatio-temporal characteristics of the secondary sources is apparent or when secondary sources are not properly positioned and orientated could not be included in the present paper.

It was shown that only a subset of the spherical harmonics coefficients of the loudspeaker directivity, namely the rotation-symmetric modes, have to be known. We assumed in the presented investigation that these directivity coefficients are precisely known. This requires high resolution measurements of the coefficients in order to assure that no considerable spatial aliasing occurs. These measurements can be assumed to be less complex than in the conventional compensation approaches which require to measure the entire loudspeaker array (refer to Sec. I). Furthermore, the fact that only rotational symmetric modes of the directivity need to measured provides potential to further reduce complexity.

It is also advisable that the radius of the microphone array which is employed in the measurement is not too different from the radius of the secondary source contour under consideration. The presented approach implicitly includes an extrapolation of the microphone array measurements to the secondary source contour. The restrictions of extrapolation of such spatially discrete data when spatial aliasing is apparent is not known.

It has to be noted that the calculation of spherical Hankel functions of high orders and large arguments (i.e. high frequencies or large radii of the secondary source contour) requires high numerical precision. The fact that the modes of the directivity filter can be pre-computed facilitates the task. However, the measurement of high modes and high frequencies of the loudspeakers' spatio-temporal transfer function is sensitive towards measurement noise and limitations arise [9] which have not been investigated in detail.

Future work includes error analysis as described above.

References

[1] E. Corteel, "Equalization in an extended area using multichannel inversion and wave field synthesis," *JAES*, vol. 54, no. 12, pp. 1140–1161, Dec. 2006.

[2] T. Betlehem and T. D. Abhayapala, "Theory and design of sound field reproduction in reverberant rooms," *JASA*, vol. 117, no. 4, pp. 2100–2111, April 2005.

[3] S. Spors, H. Buchner, R. Rabenstein, and W. Herbordt, "Active listening room compensation for massive multichannel sound reproduction systems using wave-domain adaptive filtering," *JASA*, vol. 122, no. 1, pp. 354–369, July 2007.

[4] J. Ahrens and S. Spors, "An analytical approach to sound field reproduction using circular and spherical loudspeaker distributions," *Acta Acustica utd. with Acustica*, vol. 94, no. 6, pp. 988–999, Nov./Dec. 2008.

[5] ——, "Sound field reproduction employing non-omnidirectional loudspeakers," in *126th Conv. of the AES*, Munich, Germany, May. 7–10 2009.

[6] ——, "An analytical approach to 2.5D sound field reproduction employing circular distributions of non-omnidirectional loudspeakers," in *17th European Signal Processing Conference (EUSIPCO)*, Glasgow, Scotland, August 24–28th 2009.

[7] E. G. Williams, *Fourier Acoustics: Sound Radiation and Nearfield Acoustic Holography*. London: Academic Press, 1999.

[8] J. Driscoll and D. Healy, "Computing fourier transforms and convolutions on the 2-sphere," *Advances in Applied Mathematics*, vol. 15, no. 2, pp. 202–250, June 1994.

[9] F. Fazi, V. Brunel, P. Nelson, L. Hörchens, and J. Seo, "Measurement and fourier-bessel analysis of loudspeaker radiation patterns using a spherical array of microphones," in *124th Convention of the AES*, Amsterdam, The Netherlands, May 17–20 2008.

[10] N. A. Gumerov and R. Duraiswami, *Fast Multipole Methods for the Helmholtz Equation in Three Dimensions*. Amsterdam: Elsevier, 2004.

[11] F. Zotter, A. Sontacchi, M. Noisternig, and R. Höldrich, "Capturing the radiation characteristics of the bonang barung," in *Proc. of the 3rd AAAA Congress*, Graz, Austria, Sept. 27–28 2007.

[12] F. Zotter and M. Noisternig, "Near- and farfield beamforming using spherical loudspeaker arrays," in *3rd Congress of the Alps Adria Acoustics Association*, Graz, Austria, Sept. 27th–28th 2007.